

Stochastic stability of traffic maps

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Abstract

We study ergodic properties of a family of traffic maps acting in the space of bi-infinite sequences of real numbers. The corresponding dynamics mimics the motion of vehicles in a simple traffic flow, which explains the name. Using connections to topological Markov chains we obtain nontrivial invariant measures, prove their stochastic stability, and calculate the topological entropy. Technically these results in the deterministic setting are related to the construction of measures of maximal entropy via measures uniformly distributed on periodic points of a given period, while in the random setting we directly construct (spatially) Markov invariant measures. In distinction to conventional results the limiting measures in non-lattice case are non-ergodic. Average velocity of individual “vehicles” as a function of their density and its stochastic stability is studied as well.

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1 Introduction

Speaking about stochastic stability of a dynamical system one means (see e.g. [24, 4]) that the most important statistical quantities related to the dynamics (e.g. Sinai-Bowen-Ruelle invariant measures) depend continuously on the addition of a small amount of true random noise. We study ergodic properties of a family of traffic maps acting in the space of bi-infinite sequences. The corresponding dynamics mimics the motion of vehicles in a simple traffic flow, which explains the name. Our aim is to construct nontrivial invariant measures of the traffic maps and to show that they are stable with respect to “natural” random perturbations – by which we mean that the motion of individual vehicles are performed with a probability p . Then the case $p = 1$ corresponds to the deterministic traffic map, while situations with $p < 1$ may be considered as random perturbations.

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We introduce the following notation. Under an *admissible configuration* x^t at time $t \in \mathbb{Z}_+ \cup \{0\}$ we mean an ordered countable set of particles (balls) of radius $r \geq 0$, centers of which are located at the points $x^t := (x_i^t)_{i \in \mathbb{Z}} \subset \mathbf{R} \subseteq \mathbb{R}^1$ such that

$$x_i^t + r \leq x_{i+1}^t - r.$$

The set of all admissible configurations we denote by $X = X(r, \mathbf{R})$. By $v > 0$ denote the maximal possible movement of a single particle per unit time, i.e.

$$0 \leq x_i^{t+1} - x_i^t \leq v.$$

The parameter $p \in (0, 1]$ stands for the probability of movement for individual particles. A single particle performs a totally asymmetric random walk (which explains why the processes of this sort are often called TASEP - Totally Asymmetric Exclusion Process) with jumps of size v , occurring with probability p , until its motion does not interfere with the motions of other particles.

Exclusion processes (collective random walks of countable collections of particles with hard core interactions) introduced by Frank Spitzer in 1970 appear naturally in a broad list of scientific applications starting from various models of traffic flows [20, 17, 12, 5, 6], molecular motors and protein synthesis in biology, surface growth or percolation processes in physics (see [21, 9] for a review), and up to the analysis of Young diagrams in Representation Theory [11]. Continuous time versions of these processes are reasonably well understood (see e.g. [19] for a general account and [1, 2, 13, 15] for recent results). The main difficulty in the analysis of discrete time versions of exclusion processes is that an arbitrary (infinite) number of particle interactions may happen simultaneously. To overcome this difficulty one needs to develop principally new approaches. Note that in the one-dimensional setting under consideration the basic restriction is the preservation of the order of particles, i.e. the particles can not overtake each other under dynamics.

We consider discrete time Markov processes $\pi(p, v, r, \mathbf{R})$ acting in the space of configurations $X = X(r, \mathbf{R})$ and the dynamics of individual particles in the configuration is defined by the relation

$$x_i^{t+1} = \begin{cases} \min\{x_i^t + v, x_{i+1}^t - 2r\} & \text{with probability } p \\ x_i^t & \text{with probability } 1 - p \end{cases} \quad (1.1)$$

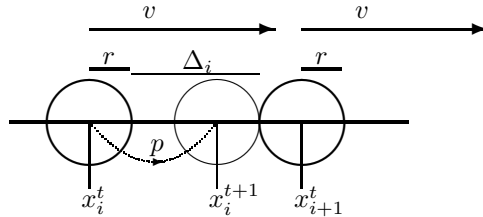


Figure 1: Exclusion process in continuum

Consider three, at first glance, very different types of exclusion processes satisfying the relation (1.1). Processes of type 1 act on a lattice $\mathbf{R} = \mathbb{Z}^1, r = 1/2, v \in \mathbb{Z}_+^1$. One lattice site may be occupied by at most one particle. Models of this type are widely used to describe the motion of vehicles on a single-lane road (see for example [20, 22]).

Processes of type 2 also act on a lattice $\mathbf{R} = \mathbb{Z}^1, v \in \mathbb{Z}_+^1$, but $r = 0$. The fundamental difference is that a lattice site can be occupied by an arbitrary number of particles. Models of this sort with a continuous-time are called zero-range processes (see, for example,

[14]) and it is convenient to use them to simulate a communication line, in which particles represent equal packets of information, waiting in queues to communication servers, located at sites of the lattice \mathbf{R} . In terms of quantum statistical mechanics processes of type 1 and 2 are related as interacting Fermi gas and free Bose gas.

Processes of type 3, which are a special case of exclusion type processes introduced in [7], act not on a lattice but on the continuous space $\mathbf{R} = \mathbb{R}^1, r \geq 0, v \in \mathbb{R}_+^1$. It will be shown (Lemma 1) that for $r = 1/2$ these processes contain all realizations of the processes of the 1st type, and for $r = 0$ all realizations of the processes the 2nd type. This allows one to obtain simultaneously analytical results for all cases. The consideration of exclusion type processes in continuum not only simplifies the analysis but due to the presence of additional symmetries (absent in lattice versions) it offers a possibility to obtain new results about the lattice cases unavailable otherwise.

A *density* of a configuration x^t (the number of particles per unit length) is defined as

$$\varrho(x^t) := \lim_{n \rightarrow \infty} n/(x_{n-1}^t - x_0^t),$$

if the latter limit makes sense.¹ As we shall see the density is the first integral of the processes under study and thus the phase space is foliated naturally by invariant subsets $X^{(\varrho)}$ consisting of admissible configurations of density ϱ .

The formula (1.1) describes the processes through dynamics of configurations of ordered particles. This is convenient for the analysis of the particles motion, but does not allow the study of invariant measures (stationary distributions). To this end we consider a modification of the process under study in which the particles are indistinguishable from each other. Speaking about invariant measures we always refer to the invariant measures of this modification. The set of probability $\pi(p, v, r, \mathbf{R})$ -invariant measures we denote by $\mathcal{M}_{p,v,r,\mathbf{R}}$.

To date, the mathematical description of invariant measures for the deterministic processes under study is essentially absent, and the only classification result [3] gives just a formal description of invariant measures for the simplest lattice process $\pi(p = 1, v = 1, r \in \{0, 1/2\}, \mathbb{Z})$ and says nothing even about the existence of nontrivial (non-atomic) invariant measures. Note that the processes $\pi(p = 1, v, r, \mathbf{R})$ possess periodic trajectories of all possible periods and hence atomic measures supported by these trajectories. In Theorems 1,4,5 for each density ϱ we prove the existence of a non-atomic invariant measure μ_ϱ supported by configurations of the given density. Moreover we give an explicit construction in terms of (spatially) Markov measures related so subshifts of finite type.

Having a large number of invariant measures it is important to distinguish “physically relevant” ones. In the case of low dimensional dynamical systems one often uses for this purpose the concept of Sinai-Bowen-Ruelle (SBR) measures. Roughly speaking the latter means that for a reasonably large family of “good” initial probabilistic measures (say all absolutely continuous measures) their images under dynamics converge in Cesaro means to the same SBR measure. In the present infinite-dimensional setting the choice of “good” initial measures is not obvious and the control over the convergence of their images is not available at present. An alternative approach consists in the analysis of stochastic stability of invariant measures. In a number of cases it has been shown (see, e.g., [4]) that SBR measures are exactly those that are stochastically stable.

¹The one-sidedness of the growth of segments (x_0^t, x_{n-1}^t) is due to the fact that all particles move in the same direction. The definition of the density in the general case is more complicated (see e.g. [7]).

As we already noted a natural choice of random perturbations for the deterministic processes $\pi(p = 1, v, r, \mathbf{R})$ are random processes $\pi(p < 1, v, r, \mathbf{R})$. Here again in the discrete time case only very partial results about invariant measures are known in the literature. Nevertheless the simplest lattice case $\pi(p < 1, v = 1, r = 1/2, \mathbb{Z})$ and especially its much simpler continuous time version was intensively studied in physics literature from this point of view (see [1, 13, 14, 15, 16, 20, 22, 23]). In particular, in [22, 23] using a mean field approximation and an interesting and nontrivial combinatorial argument (apparently not quite complete without the exact analysis of the limit construction) the authors have found necessary conditions (equivalent to our formula (4.2) for the existence of nontrivial invariant measures. On the other hand, the next step – the proof that the constructed measures are indeed invariant under dynamics was not explicitly addressed. The construction proposed in the proof of Lemma 4 provides not only a complete proof of this statement but also in its first few lines gives the crucial formula (4.2) without any limit transitions. By means of a completely different approach M.Kanai [18] has studied the same process on a finite discrete ring (instead of the infinite lattice) and obtained an exact rather complicated formula for the invariant measure as a function of the ring's length L . When L goes to infinity this invariant measure converges to the limit derived in [22], which is of independent interest despite a number of restrictions assumed in this paper. The approaches elaborated in [22, 23, 18] use in one way or another properties of invariant measures of the process $\pi(p, v = 1, r = 1/2, \{1, 2, \dots, L\})$ on a finite ring. Actually in [22, 23] the Markov structure of the invariant measure was discovered (without the proof of its existence) while in [18] the unique invariant measure is no longer Markovian and becomes Markovian only in the limit as $L \rightarrow \infty$. A coupling construction proposed by L. Gray [16] allows one to show that for a fixed density ϱ the Markov invariant measure is ergodic and unique among translationally invariant measures.

It is worth noting that the above mentioned lattice constructions cannot be extended to the case of long jumps ($v > 1$). To overcome this difficulty we first study the continuous setting to obtain the invariant measure for the process $\pi(p < 1, v = 1, r = 1/2, \mathbb{R})$ and then extend this result for all values of v, r using a wide set of symmetries available in \mathbb{R} (but not in \mathbb{Z}). After that we are able to return back to the lattice cases.

Despite the fact that the constructions of invariant measures in deterministic and stochastic settings are rather different and neither of them can be applied in another setting, the proof of stochastic stability is relatively simple and is based on the explicit description of the invariant measures.

Our main result about invariant measures is as follows.

Theorem 1 *The process $\pi(p, v, r, \mathbb{R})$ for each $0 < p \leq 1, v > 0, r \geq 0$ and each particle density $\varrho \in (0, 1/(2r)]$ possesses a nontrivial invariant measure μ_ϱ^p supported by the set of configurations of density ϱ and $\mu_\varrho^p \xrightarrow{p \rightarrow 1} \mu_\varrho^1$ in the weak sense.*

One of the most important characteristics of the processes under consideration is an *average velocity of particles* during time $t > 0$:

$$V(x, i, t) := (x_i^t - x_i^0)/t.$$

In [7] it was shown that under very general assumptions (including, in particular, dynamics of particles moving in opposite directions), which definitely hold true for all processes considered in the present work, the a.s. limit of the statistics $V(x, i, t)$ as $t \rightarrow \infty$ (if it

exists) depends only on the density ϱ of the configuration x (see Theorem 3). Therefore, to calculate the average velocity

$$V(x) = V(\varrho(x)) := \lim_{t \rightarrow \infty} V(x, i, t)$$

(where the convergence is considered in the almost sure sense) for a configuration of a given density it is enough to determine the latter for a specially selected initial configuration of the same density (for which the calculation is convenient). Note that the existence of the limit under question, not to mention the explicit formulas for it, previously has been proven only in the deterministic setting (i.e. for $p = 1$), see [7]. The following result gives explicit relations between the average velocity and other parameters of the process $\pi(p, v, r, \mathbb{R})$.

Theorem 2 *For the process $\pi(p, v, r, \mathbb{R})$ $\forall v, \varrho \in \mathbb{R}_+^1$, $0 \leq r < 1/(2\varrho)$, $p \in (0, 1)$ with any admissible initial configuration of density ϱ the average velocity $V(\varrho, r)$ is well defined and is calculated as follows:*

$$V(\varrho, r) = (1 - 2r\varrho) V\left(\frac{\varrho}{1 - 2r\varrho}, 0\right), \quad (1.2)$$

$$V(\varrho, 0) = \left[1 + v\varrho - \sqrt{(1 + v\varrho)^2 - 4pv\varrho}\right] / (2\varrho) \xrightarrow{p \rightarrow 1} \min\{1/\varrho, v\}. \quad (1.3)$$

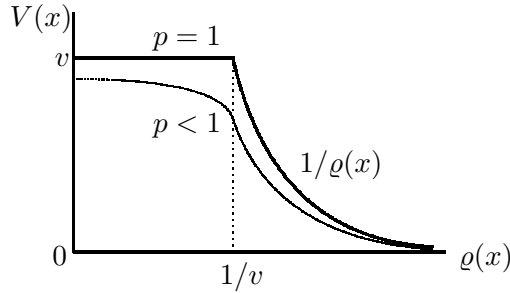


Figure 2: Fundamental diagrams (average velocity against density) for the process $\pi(p, v, r = 0, \mathbb{R})$.

In physics literature results similar to Theorem 2 are known for the simplest lattice TASEP process $\pi(p < 1, v = 1, r = 1/2, \mathbb{Z})$ (see [22, 23, 18]). Here again one needs to make comments about missing arguments of the same sort as in the analysis of invariant measures plus that the result should hold not only for “typical” initial configurations (see discussion below and in Section 5).

The dynamical coupling construction developed in [7, 8] allows one to get complete information about the properties of average velocities in the deterministic setting (i.e. for $p = 1$). This construction does not require the study of (numerous) invariant measures of the process, but gives only conditional (upon their existence) results in the stochastic case (albeit under much broader assumptions about the process: the local velocities $v = v_i$ are iid random variables). In the stochastic case one cannot avoid the analysis of invariant measures. Explicit expression for the invariant measure μ allows one to derive (by Birkhoff’s ergodic theorem) the formula for the average velocity for μ -a.a. initial configurations. To extend this result to a much more broad setting (formulated in Theorem 2)

for all particle configurations for which densities are well defined, one needs to use metric properties of the process obtained earlier. Namely in [7] it has been shown that for all particle configurations of a given density the average velocity is the same. Therefore it is enough to calculate this statistics for a single suitable initial configuration (e.g. a configuration typical with respect to the measure μ).

The paper is organized as follows. In Section 2 we discuss connections between the processes under study and briefly review known results related to particle densities and average velocities. Ergodic properties of the deterministic traffic map will be discussed in Section 3, which we shall finish by the calculation of the topological entropy of the traffic map in continuum. The true random setting (i.e. $0 < p < 1$) will be analyzed in Section 4. In Section 5 we shall study average particle velocities, obtain explicit relations between these statistics and parameters of the process, and prove their stochastic stability. Finally in Section 5.4 we shall discuss heterogeneous versions of the processes under consideration (particles with different sizes in the same configuration and presence of randomly distributed static obstacles in space).

2 Metric properties

Here we shall discuss connections between the processes under study and briefly review known results related to particle densities and average velocities.

Let $\Delta_i = \Delta_i(x^t) := x_{i+1}^t - x_i^t - 2r$ stands for the distance between boundaries of the balls corresponding to subsequent particles in the configuration x^t (see Fig. 1). We refer to Δ_i as the *gap* corresponding to the i -th particle in x^t . If $\Delta_i(x^t, z) < v$ we say that the i -th particle is *blocked* (meaning that its motion is blocked by the $i + 1$ -th particle) at time t and *free* otherwise. By a *cluster* of particles in a configuration $x^t \in X$ we mean a locally maximal collection of consecutive blocked particles.

To emphasize the dependence of various statistics on the parameter $r \geq 0$ (ball's radius) we denote by $x(r)$ and $X(r, \mathbf{R})$ the configurations with balls of radius r and the corresponding space of admissible configurations. In the special case of point-particles ($r = 0$) the dependence on the radius will be omitted.

We say that two particle processes whose dynamics is described by the relation (1.1) are *statically coupled* if all random choices related to the motion of particles with the same indices in these processes coincide.

The following results show relations between the three types of exclusion processes (two lattice and one in continuum) introduced in the previous Section.

Lemma 1 *For any given v, p the following relations are valid:*

1. $\pi(p, v \in \mathbb{Z}_+, r = 1/2, \mathbb{Z}) = \pi(p, v, r = 1/2, \mathbb{R})$ if $x^0(r = 1/2) \subset \mathbb{Z}$;
2. $\pi(p, v \in \mathbb{Z}_+, r = 0, \mathbb{Z}) = \pi(p, v, r = 0, \mathbb{R})$ if $x^0(r = 0) \subset \mathbb{Z}$;
3. $\forall r > 0$ there exists an affine homeomorphism $\varphi = \varphi_r : X(r, \mathbb{R}) \rightarrow X(0, \mathbb{R})$, such that

$$\varphi \circ \pi(p, v, r, \mathbb{R}) = \pi(p, v, r = 0, \mathbb{R}) \circ \varphi;$$

4. $\forall u, v > 0, 0 < p \leq 1$ a sub-lattice $\mathbf{R}_{u,v} := v\mathbb{Z} + u$ is invariant with respect to the process $\pi(p, v, r = 0, \mathbb{R})$, i.e. $x^0 \subset \mathbf{R}_{u,v} \Rightarrow x^t \subset \mathbf{R}_{u,v} \forall t > 0$.

Proof. The only nontrivial statement here is the item (3). Observe that any two particle configurations $x(r)$, $\acute{x}(\acute{r})$ having the same sequence of gaps $\Delta := \{\Delta_i\}$ may be transformed to each other by a one-to-one map

$$\acute{x}_i(\acute{r}) = \varphi_r(x_i(r)) := x_i(r) - 2i(r - \acute{r}) \quad \forall i \in \mathbb{Z}. \quad (2.1)$$

This allows one to choose the homeomorphism φ_r as follows

$$(\varphi_r(x(r)))_i := x_i(r) - 2ir \quad \forall i \in \mathbb{Z}.$$

Now the conjugation between the processes having particles of different sizes follows from the equality between sequences of gaps Δ^t between particles in the statically coupled processes $\pi(p, v, r, \mathbb{R})$ and $\pi(p, v, 0, \mathbb{R})$. \square

The correspondence between densities and average particle velocities for statically coupled processes with configurations consisting of balls of different radiuses $r \neq \acute{r} \geq 0$ is summarized as follows.

Lemma 2 [7]

1. The density $\varrho(x^t)$ is preserved by dynamics, i.e. $\varrho(x^t) = \varrho(x^{t+1}) \quad \forall t \geq 0$.
2. Let configurations $x(r) \in X(r, \mathbb{R})$, $r > 0$ and $\acute{x}(\acute{r}) \in X(\acute{r}, \mathbb{R})$, $\acute{r} \geq 0$ have the same sequence of gaps $\{\Delta_i\}$. Then

$$\varrho(x(r)) = \frac{\varrho(\acute{x}(\acute{r}))}{1 + 2(r - \acute{r})\varrho(\acute{x}(\acute{r}))}.$$

3. Let additionally the processes $x^t(r)$ and $\acute{x}^t(\acute{r})$ be statically coupled. Then $\forall i, t$

$$V(x(r), i, t) = V(\acute{x}(\acute{r}), i, t).$$

Results obtained in [7] allow also to claim the coincidence of average velocities both for individual particles and for configurations having equal densities.

Theorem 3 ([7]) For the process $\pi(p, v, r, \mathbf{R}) \quad \forall v, r \in \mathbf{R}, \varrho > 0, p \in (0, 1]$ the following claims are valid. Let $x, y \in X(r, \mathbf{R})$, $\varrho(x) = \varrho(y) = \varrho$ and let the average velocity $V(y)$ be well defined. Then

$$|V(x, i, t) - V(y, j, t)| \xrightarrow{t \rightarrow 0} 0$$

for any $i, j \in \mathbb{Z}$.

3 Ergodic properties: deterministic case (p=1)

Despite the abundance of invariant measures supported by time-periodic trajectories (existing for all periods) nothing was known previously about the existence of nontrivial invariant measures. Our main result in this direction in the case $\mathbf{R} := \mathbb{R}$ is the following claim.

Theorem 4 *The deterministic process $\pi(p = 1, v, r, \mathbb{R})$ for each $v > 0, r \geq 0$ and each particle density $\varrho \in (0, 1/(2r)]$ possesses a non-atomic invariant measure μ_ϱ such that $E_{\mu_\varrho}[x_i^t] = \varrho$.*

The proof of this result will be given step by step in Sections 3.1, 3.2. In the simplest lattice case the construction of the invariant measure will be described explicitly in terms of the so called (spatially) Markov measures. The notion *Markov measure* here refers to the unique invariant measure for the Markov shift on $\{0, 1\}^{\mathbb{Z}}$ with a non-degenerate transition matrix (p_{ij}) , $i, j \in \{0, 1\}$, $p_{ij} \geq 0$, $\sum_j p_{ij} = 1$. In terms of configurations of particles, 1 corresponds to the presence of a particle at a site of the lattice \mathbb{Z} , and 0 - its absence.

It is worthwhile giving an equivalent definition of the Markov measure without any connection to random processes (see e.g. [25]). A measure μ on $\{0, 1\}^{\mathbb{Z}}$ is Markov iff for any cylinder² with the base $[AbC]$, where A and B two arbitrary finite binary sequences while b is a single binary letter, we have

$$\mu([b]) \cdot \mu([AbC]) = \mu([Ab]) \cdot \mu([bC]). \quad (3.1)$$

Note that [25] claims that the Markov measures appear extremely rarely in cellular automata systems and thus their existence for traffic maps turns out to be a surprise both in deterministic and random settings.

3.1 Lattice traffic map

Theorem 5 *The deterministic lattice process $\pi(p = 1, v = 1, r = 1/2, \mathbb{Z})$ possesses a Markov invariant measure μ , being a weighted sum of two measures of maximal entropy for shift-maps acting in opposite directions. Besides, there is a 1-parameter family of invariant measures $\{\mu_{1,1,1/2,\mathbb{Z},\varrho}\}_\varrho$ such that $E_{\mu_{1,1,1/2,\mathbb{Z},\varrho}}[x_i^t] = \varrho$.*

Note that the construction of Markov invariant measures which we shall elaborate for the analysis of the random setting in Section 4 cannot be applied here and we shall use a very different strategy related to the construction of measures of maximal entropy well known in theory of low-dimensional hyperbolic dynamical systems (see e.g. [10]).

Let $\mathbf{B} := \{0, 1\}^{\mathbb{Z}}$ be the space of binary sequences and let \mathbf{B}_\pm be its two subspaces defined by the transition matrices $M_+ := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $M_- := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, namely

$$\mathbf{B}_+ := \{b \in \mathbf{B} : b_i + b_{i+1} \neq 2 \ \forall i \in \mathbb{Z}\},$$

$$\mathbf{B}_- := \{b \in \mathbf{B} : b_i + b_{i+1} \neq 0 \ \forall i \in \mathbb{Z}\}.$$

Consider two shift-maps $\sigma_\pm : \mathbf{B} \rightarrow \mathbf{B}$ defined as follows:

$$(\sigma_+ b)_i := b_{i-1}, \quad (\sigma_- b)_i := b_{i+1},$$

i.e. these two maps shift a sequence by one position in opposite directions. Obviously

$$\sigma_\pm \mathbf{B}_\pm \equiv \mathbf{B}_\pm.$$

²a set of all sequences taking a given collection of values at a given collection of indices, called its base

Denote by $T : \mathbf{B} \rightarrow \mathbf{B}$ the map corresponding to the deterministic process $\pi(p = 1, v = 1, r = 1/2, \mathbb{Z})$. Recall that the density of a binary sequence $b \in \mathbf{B}$ is defined exactly as the density of a configuration of ones in b given in Section 1.

Theorem 5 claims only the existence of Markov invariant measures but in fact we shall prove that the restriction of the dynamical system (T, \mathbf{B}) to the Cantor sets \mathbf{B}_\pm possesses *massive* invariant measures in the sense that these measures are positive on each open subset. Before to give the proof of this result let us discuss reasons for the absence of massive invariant measures in the entire space. In Section 4 we shall show that in the true random case ($0 < p < 1$) for each $a = p_{01} \in (0, 1)$ there is a nontrivial measure $\mu_a^{(p)}$ (constructed in the proof of Theorem 7) whose value on the cylinder with the base $[1100]$ is equal to

$$\mu_a^{(1)}([1100]) = p_1 p_{11} p_{10} p_{00} = p_1 (1 - p_{10}) p_{10} p_{00}.$$

On the other hand, by (4.4)

$$p_{10} = (1 - p_{01}) / (1 - p p_{01}) \xrightarrow{p \rightarrow 1} 1.$$

Therefore for each p_{01} we have

$$\mu_a^{(1)}([1100]) \xrightarrow{p \rightarrow 1} 0,$$

and thus the limiting (as $p \rightarrow 1$) measure cannot be massive. This explains why in the deterministic case when $p = 1$ one should not expect the existence of true massive invariant measures. We conjecture that for each T -invariant probability measure η there exists an open subset $B \in \mathbf{B}$ with $\eta(B) = 0$. However at present we do not have a complete proof of this statement.

Proof of Theorem 5. Let us start with a recipe of the construction of measures of maximal entropy for the shift-map $\sigma : \mathbf{B}_M \rightarrow \mathbf{B}_M$ with an irreducible transition matrix $M = (m_{ij})_{i,j \in \{0,1\}}$ (see e.g. [10]). For each $n \in \mathbb{Z}_+$ denote by μ_n the probability measure uniformly distributed on points of period n of the map σ . Then the measures μ_n weakly converge as $n \rightarrow \infty$ to a massive (on \mathbf{B}_M) probability measure μ_σ . The latter coincides with the unique invariant distribution for the stationary Markov chain with the transition probability matrix

$$P_\sigma := (m_{ij} m_j / (\lambda_M m_i)),$$

where λ_M is the maximal eigenvalue of the matrix M and (m_i) is the corresponding eigenvector. The measure μ_σ represents the only invariant measure maximizing the metric entropy of the dynamical system (σ, \mathbf{B}_M) with the transition matrix M , which explains the reason for its name. Calculating the leading elements of the spectra of the matrices M_\pm we get

$$\lambda_+ = \lambda_- =: \lambda := (1 + \sqrt{5})/2, \quad m_0^+ = m_1^- = 1/\lambda, \quad m_1^+ = m_0^- = 1 - 1/\lambda.$$

Despite the absence of a similar result for the dynamical system (T, \mathbf{B}) we try to follow this recipe. A serious problem here is that we do not have good control over all time-periodic points of a given period for the map T . To overcome this difficulty we select a subset of time-periodic points with which we shall work.

Observe that if a configuration $b \in \mathbf{B}$ is spatially periodic then $\forall t \in \mathbb{Z}_+ T^t b$ is spatially periodic with the same period. This fact follows immediately from the definition of the

map T and is discussed in detail in [5]. Additionally in [5] it has been shown that if either $\sigma_+^n b = b$ and $b \in \mathbf{B}_+$ or $\sigma_-^n b = b$ and $b \in \mathbf{B}_-$ then this spatially periodic point is time-periodic for the map T with the same period n . Note that $\varrho(b) \leq 1/2$ if $b \in \mathbf{B}_+$ and $\varrho(b) \geq 1/2$ if $b \in \mathbf{B}_-$.

The absence of non spatially periodic time-periodic points of the map T would complete the description of time-periodic points. Unfortunately this is not the case. The point is that the map is not one-to-one. Using this let us construct a sketch of a counter-example. Let b be spatially and time periodic for the map T with the same period n , i.e. $b_i = b_{i+n} \forall i \in \mathbb{Z}$ and $T^n b = b$. Thus a finite word $(b_n, b_{-n+1}, \dots, b_{-1})$ is a pre-image of the word $(b_0, b_1, \dots, b_{n-1})$ under the action of the map T^n . Since the map T is not bijective there exists another pre-image $(b'_n, b'_{-n+1}, \dots, b'_{-1})$ of the word $(b_0, b_1, \dots, b_{n-1})$. Similarly we consider a pre-image of the word $(b'_n, b'_{-n+1}, \dots, b'_{-1})$ under the action of the map T^n and denote it by $(b'_{-2n}, b'_{-2n+1}, \dots, b'_{-n-1})$. Continuing this procedure we are getting a point $b' := (\dots, b'_{-2}, b'_{-1}, b_0, b_1, b_2, \dots) \in \mathbf{B}$ such that b' is no longer spatially periodic but $T^n b' = b'$.

Nevertheless we can use a spatially periodic part of time-periodic points for our construction. For each $n \in \mathbb{Z}_+$ consider a probability measure μ_n^* uniformly distributed on spatially and time periodic points of period n of the map T . As we already noted each spatially and time periodic point of the map T is spatially periodic point of one of the shift-maps σ_\pm . Due to the symmetry of the motion of ones and zeros under the action of the map T the number C_n^+ of n -periodic points of density less or equal to $1/2$ differs at most by n from the number C_n^- of n -periodic points of density larger than $1/2$, while C_n^\pm are of order n^λ (see the calculation of the exponent $\lambda > 0$ below). Therefore one can represent the measure μ_n^* as follows

$$\mu_n^* = (C_n^+ \mu_n^+ + C_n^- \mu_n^-) / (C_n^+ + C_n^-),$$

where μ_n^\pm are probability measures uniformly distributed on n -periodic points of the shift-maps σ_\pm . Being uniformly distributed on time-periodic trajectories the measure μ_n^* is T -invariant for each n .

We already know that the measures μ_n^\pm converge as $n \rightarrow \infty$ to the corresponding measures of maximal entropy μ_σ^\pm for the shift-maps σ_\pm . Therefore

$$\mu_n^* \xrightarrow{n \rightarrow \infty} (\mu_\sigma^+ + \mu_\sigma^-) / 2 =: \mu_T.$$

The weak massive property for the limit measure follows from the similar statement for the measures μ_σ^\pm on the sets \mathbf{B}_\pm . Indeed by the construction for each cylinder on \mathbf{B}_+ its μ_σ^+ measure is positive, and the similar statement holds for each cylinder on \mathbf{B}_- and the measure μ_σ^- .

Note that according to our construction all three limit measures μ_T, μ_σ^\pm are T -invariant. Moreover, the equality of the leading eigenvalues $\lambda_+ = \lambda_- =: \lambda := (1 + \sqrt{5})/2$ implies the coincidence of the corresponding metric entropies (being equal to $\ln \lambda$).

It remains to construct for each $\varrho \in [0, 1]$ the T -invariant measure μ_ϱ supported by the configurations of density ϱ . This can be done using either Gibbsian reconstruction of the measures μ_σ^\pm or by an explicit representation in terms of Markov shifts. Let us discuss the latter approach.

The action of a shift-map σ is equivalent to the time-shift along realizations of the Markov chain with the transition probability matrix compatible with the corresponding

binary transition matrix M . Recall that matrices $A = (a_{ij})$ and $B = (b_{ij})$ with nonnegative entries are *compatible* if $a_{ij}b_{ij} = 0$ implies $a_{ij} + b_{ij} = 0$.

A probability transition matrix compatible with the matrix M_+ is written as $P_+ := \begin{pmatrix} 1-a & a \\ 1 & 0 \end{pmatrix}$ with a single parameter $0 \leq a \leq 1$. The function $\varrho := a/(1+a)$ defines a bijection between the values of the parameter a and the set of particle densities ϱ . Thus for each density $\varrho \in [0, 1/2]$ we obtain a massive (spatially) Markov T -invariant measure μ_ϱ .

Similarly one considers the matrix M_- which allows one to construct massive invariant measures μ_ϱ with $\varrho \in (1/2, 1]$. \square

Theorem 5 gives a complete recipe for the construction of nontrivial invariant measures for the process $\pi(p = 1, v = 1, r = 1/2, \mathbb{Z})$.

3.2 Existence of massive invariant measures in continuum

In this Section we develop a special machinery extending the measures μ_ϱ (constructed for the lattice case) first to nontrivial invariant measures of the deterministic process $\pi(p = 1, v = 1, r = 1/2, \mathbb{R})$ acting on the real line and then to deterministic processes $\pi(p = 1, v > 0, r \geq 0, \mathbb{R})$ with arbitrary $v > 0, r \geq 0$.

Recall that in this more complicated ‘continuous’ setting a finite *cylinder* with the base defined by a finite subset of integers I and a collection $C := \{C_i\}_{i \in I}$ of open intervals³ is the subset $\mathcal{C}_{I,C} := \{x \in X : x_i \in C_i \ \forall i \in I\}$. We endow the space of admissible configurations X with the σ -algebra \mathcal{B} generated by the finite cylinders defining a topology in this space.

By Lemma 1(1) the measures $\mu_\varrho = \mu_{\varrho,r}$ are $\pi(p = 1, v = 1, r = 1/2, \mathbb{R})$ -invariant. For each $r' \geq 0$ applying the affine transformation (2.1), obtained in Lemma 1(3), to the measure μ_ϱ we are getting a new probability measure $\mu_{\varrho',r'}$ with $\varrho' := \varrho/(1 + 2(1/2 - r')\varrho)$, which is $\pi(p = 1, v = 1, r', \mathbb{R})$ -invariant. Since ϱ takes all values from the interval $[0, 1]$ the new variable ϱ' takes all values from the interval $[0, 1/(2r')]$.

Making yet another spatial change of variables $z \rightarrow vz + w$ with parameters $v > 0, w \geq 0$ we are obtaining from the measure $\mu_{\varrho',r'}$ a two-parameter family of probability measures $\mu(p = 1, \varrho'', r'', v, w)$ supported on configurations of balls of radius $r'' := vr'$ and having density $\varrho'' := \varrho'/v$. Applying again Lemma 1(4) we see that for each ϱ'', r', v, w the measure $\mu(p = 1, \varrho'', r'', v, w)$ is $\pi(p = 1, v, r'', \mathbb{R})$ -invariant.

On the other hand, these measures cannot be massive, since they are supported by very thin sets. To construct a massive (albeit non-ergodic) invariant measure from this family we consider a measure $\mu(p = 1, \varrho'', r'', v)$ having marginals $\mu(p = 1, \varrho'', r'', v, w)$ on sub-lattices $v\mathbb{Z} + w$ and uniformly distributed with respect to the parameter $w \in [0, v]$.

This finishes the proof of the existence of massive invariant measures for the deterministic traffic map in continuum.

The construction of the massive invariant measure in the lattice case ($\mathbf{R} := \mathbb{Z}$) with general $v \in \mathbb{Z}_+$ is similar except that $w \in \{0, 1, \dots, v-1\}$ and $r'' = 1/2$ or $r'' = 0$ to be able to work with point particles in lattice setting.

³In general the cylinder $\mathcal{C}_{I,C}$ might be empty for nonempty sets I, C .

3.3 Entropy

To finalize the description of ergodic properties of the processes under study in the pure deterministic setting we show (following the approach developed in [7]) that these processes are strongly chaotic. Our dynamical system is defined by a deterministic map $T_v : X \rightarrow X$ from the set of admissible configurations into itself. Our aim is to show that the topological entropy of this map is infinite.⁴

We refer the reader to [26, 27] for detailed definitions of the topological and metric entropies for deterministic dynamical systems and their properties that we use here. To avoid difficulties related to the non-compactness of the phase space we define the topological entropy of a map T_v (notation $h_{\text{top}}(T_v)$) as the supremum of metric entropies of this map taken over all probabilistic invariant measures (compare to the conventional definition of the topological entropy and its properties discussed, e.g. in [27]).

We start the analysis with the action of a shift-map in continuum $\sigma_v : X \rightarrow X$ defined as

$$(\sigma_v x)_i := x_i + v \quad i \in \mathbb{Z}, \quad x \in X, \quad v > 0.$$

Lemma 3 *The topological entropy of the shift-map in continuum σ_v is infinite.*

Proof. The continuity of the shift-map in continuum in the topology induced by the σ -algebra \mathcal{B} generated by finite cylinders is implied by the fact that a preimage of a finite cylinder under the action of σ_v is again a finite cylinder.

The idea of the calculation of the entropy is to construct an invariant subset of X on which the map σ_v is isomorphic to the full shift-map in the space of sequences with a countable alphabet. The result follows from the observation that the topological entropy of the full shift-map $\sigma^{(n)}$ with the alphabet consisting of n elements is equal to $\ln n$ (see, e.g. [26, 27]).

Let $\alpha := \{\alpha_i\}_{i \in \mathbb{Z}_+}$ with $\alpha_i \in (0, v)$ and let $\alpha^{(n)} := \{\alpha_i\}_{i=1}^n$. Consider a sequence of subsets $X^{(n)} \subset X$ consisting of *all* configurations $x \in X$ satisfying the condition

$$x_{2k} \in v\mathbb{Z}, \quad x_{2k+1} \in x_{2k} + \alpha^{(n)} \quad \forall k \in \mathbb{Z}.$$

Then $X^{(n)}$ is σ_v -invariant and the restriction $\sigma_v|_{X^{(n)}}$ is isomorphic to the full shift-map $\sigma^{(n)}$ with the alphabet A^n consisting of n elements $\{a_i\}$ of type $a_i := \{[0, \alpha_i), [\alpha_i, v)\}$, i.e. each element is represented by a pair of neighboring intervals. Therefore the topological entropy of $\sigma^{(n)}$ is equal to $\ln n \xrightarrow{n \rightarrow \infty} \infty$. \square

Theorem 6 *The topological entropy of the traffic map in continuum \mathbb{R} is infinite.*

Proof. The traffic map is continuous in the topology induced by the σ -algebra \mathcal{B} generated by finite cylinders by the same argument as in the case of the shift-map.

Observe that the subset

$$X_{>v} := \{x \in X : \Delta_i(x) \geq v \quad \forall i \in \mathbb{Z}\}$$

⁴Normally one says that a map is chaotic if its topological entropy is positive, so infinite value of the entropy indicates a very high level of chaoticity.

of the set of admissible configurations is T_v -invariant. Therefore

$$h_{\text{top}}(T_v) \geq h_{\text{top}}(T_v|X_{>v})$$

and for our purposes it is enough to show that the latter is infinite. On the other hand, by the definition of the map T_v we have $T_v|X_{>v} \equiv \sigma_v|X_{>v}$.

One cannot apply the result of Lemma 3 directly because in the case under consideration the gaps between particles are greater or equal to v by the construction, while in the proof of Lemma 3 the gaps did not exceed v . To this end one sets $\alpha_i \in (v, 2v)$ and modifies the definition of $X^{(n)}$ as follows:

$$\tilde{X}^{(n)} := \{x_{2k} \in 3v\mathbb{Z}, \quad x_{2k+1} \in x_{2k} + \alpha^{(n)} \quad \forall k \in \mathbb{Z}\}.$$

Consider the the alphabet $A^{(n)}$ with elements of type $a_i := \{[0, \alpha_i), [\alpha_i, 3v)\}$, $\alpha_i \in \alpha^{(n)}$. Then the 3-d power of the map $T_v|X_{>v}$ is isomorphic to the full shift-map $\sigma^{(n)}$ with the alphabet $A^{(n)}$. Using that

$$3h_{\text{top}}(T_v|X_{>v}) = h_{\text{top}}((T_v|X_{>v})^3) = h_{\text{top}}(\sigma^{(n)}) = \ln n$$

we get the result. \square

4 Ergodic properties: stochastic case ($0 < p < 1$)

4.1 The simplest lattice process $\pi(p, v = 1, r = 1/2, \mathbb{Z})$

We start with the construction of a nontrivial Markov invariant measure for the simplest lattice TASEP.

Lemma 4 *Let $P := \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}$ be a probability matrix with positive entries and the left leading normalized eigenvector (p_0, p_1) . Then the measure μ on X , defined on cylinders by the relation*

$$\mu([a_1, a_2, \dots, a_n]) := p_{a_1} \prod_{i=1}^{n-1} p_{a_i, a_{i+1}}, \quad a_i \in \{0, 1\}, \quad (4.1)$$

is invariant with respect to the process $\pi(p, v = 1, r = 1/2, \mathbb{Z})$ iff

$$p_{00}p_{11} = (1 - p)p_{10}p_{01}. \quad (4.2)$$

Proof. For a Markov chain π acting on a space of binary sequences we say that a cylinder B is a (partial) *pre-image* of a cylinder A if the probability $\Pr(\pi(B) = A) > 0$. We need to check that for any cylinder its measure is equal to the sum of measures of all its pre-images multiplied by the corresponding transition probabilities. The proof follows by induction on the cylinder's length n .

We start with cylinders of length 1 and 2. The following tables show all pre-images of cylinders of length 1 and 2. The left column corresponds to cylinders and their pre-images, while the right column shows stationary probabilities for the cylinder (the 1st line) and

stationary probabilities for its pre-images multiplied by the transition probabilities. Here $\bar{p} := 1 - p$ is the probability that the jump does not take place.

0	p_0	1	p_1	10	$p_1 p_{10}$	11	$p_1 p_{11}$
00	$p_0 p_{00}$	10	$\bar{p} p_1 p_{10}$	10	$\bar{p} p_1 p_{10}$	110	$\bar{p} p_1 p_{11} p_{10}$
10	$\bar{p} p_1 p_{10}$	11	$p_1 p_{11}$	110	$p p_1 p_{11} p_{10}$	111	$p_1 p_{11}^2$
10	$p p_1 p_{10}$	10	$p p_1 p_{10}$	100	$p p_1 p_{10} p_{00}$	1010	$p \bar{p} p_1 p_{10}^2 p_{01}$
				1010	$p^2 p_1 p_{10}^2 p_{01}$	1011	$p p_1 p_{10} p_{01} p_{11}$

00	$p_0 p_{00}$	01	$p_0 p_{01} = p_1 p_{10}$
000	$p_0 p_{00}^2$	10	$p p_1 p_{10}$
100	$\bar{p} p_1 p_{10} p_{00} = \bar{p} p_0 p_{01} p_{00}$	0010	$\bar{p} p_0 p_{00} p_{01} p_{10} = \bar{p} p_1 p_{10}^2 p_{00}$
0010	$p p_0 p_{00} p_{01} p_{10} = p p_1 p_{10}^2 p_{00}$	0011	$p_0 p_{00} p_{01} p_{11} = \bar{p} p_1 p_{10}^2 p_{01}$
1010	$\bar{p} p p_1 p_{10}^2 p_{01} = \bar{p} p p_0 p_{01}^2 p_{10}$	1010	$\bar{p}^2 p_1 p_{10}^2 p_{01} = \bar{p} p_1 p_{10} p_{00} p_{11}$
		1011	$\bar{p} p_1 p_{10} p_{01} p_{11}$

In the first two cases the equivalence of the probability in the 1st line to the sum of other probabilities is trivial. In the 3d case assuming that $p p_{10} \neq 0$ we get the following condition for the equivalence

$$1 = \bar{p} + p p_{11} + p p_{00} + p^2 p_{10} p_{01} \implies 1 = p_{00} + p_{11} + p p_{10} p_{01}, \quad (4.3)$$

while in the 4th case assuming that $p_1 p_{11} \neq 0$ we have

$$1 = \bar{p} p_{10} + p_{11} + p \bar{p} p_{10}^2 p_{01} / p_{11} + p p_{10} p_{01} \implies p_{00} p_{11} = \bar{p} p_{10} p_{01}.$$

The last calculation proves that the assumption (4.2) is necessary. A direct calculation shows that despite appearances the relation (4.3) is equivalent to (4.2).

The checking of the cases 5 and 6 can be done similarly using the property $p_0 p_{01} = p_1 p_{10}$ (which we already used in the corresponding tables).

Assume now that the claim is already proven for all cylinders with bases less or equal to $n > 1$. To reduce the case of length $n + 1$ to n observe that the only difference between smaller cylinder's length to the larger one consists of an additional letter 0 or 1 at the right end of the cylinder's base. Therefore it is enough to show that the corresponding probability changes by p_{ab} , where a is the last letter in the shorter cylinder and b is the new letter. To prove this we consider all 4 possibilities:

$$1 \rightarrow 10, 1 \rightarrow 11, 0 \rightarrow 00, 0 \rightarrow 01.$$

The 2nd and 3d situations are relatively simple:

$$1 \rightarrow 11 : \quad 10 \rightarrow 110, 11 \rightarrow 111 \text{ and } 10 \rightarrow 1010 + 1011$$

$$p p_1 p_{10} (\bar{p} p_{10} p_{01} + p_{01} p_{11}) = p p_1 p_{10} (p_{00} p_{11} + p_{01} p_{11}) = p p_1 p_{10} p_{11}.$$

$$0 \rightarrow 00 : \quad 00 \rightarrow 000, 10 \rightarrow 100 \text{ and } 10 \rightarrow 0010 + 1010$$

(here and in the sequel “.” stands for an arbitrary symbol)

$$p p_1 p_{10} (p_{10} p_{00} + \bar{p} p_{01} p_{10}) = p p_1 p_{10} (p_{10} p_{00} + p_{00} p_{11}) = p p_1 p_{10} p_{00}.$$

Note that we use that the transition probabilities depend only on the previous letter.

In the 1st situation we proceed as follows

$$1 \rightarrow 10 : \quad 10 + 11 \rightarrow 10 + 110, \quad 10 \rightarrow 100 + 1010.$$

Denoting the product of the measure of a cylinder A and the corresponding transition probability by $K(A)$ we get

$$\begin{aligned} K([10]) + K([11]) &= p_1(\bar{p}p_{10} + p_{11}) = p(p_{10} - pp_{10} + p_{11}) = p_1(1 - pp_{10}) \\ K([10]) + K([110]) &= p_1p_{10}(\bar{p} + pp_{11}) = p_1p_{10}(1 - p + pp_{11}) \\ &= p_1p_{10}(1 - p(1 - p_{11})) = p_1p_{10}(1 - pp_{10}). \end{aligned}$$

Thus the 2nd sum differs from the 1st by the desired multiplier p_{10} .

The 4th situation is a bit more complicated. In this case we split the 3d pre-image .10 of 0 into two parts 010 and 110 giving contributions

$$K([010]) = pp_0p_{01}p_{10}, \quad K([110]) = pp_1p_{11}p_{10}$$

respectively (observe that $pp_0p_{01}p_{10} + pp_1p_{11}p_{10} = pp_1p_{10}$), and the 1st pre-image 10 of 01 also splits into two parts 010 and 110 giving contributions

$$K([010]) = pp_0p_{01}p_{10} = pp_1p_{10}^2, \quad K([110]) = pp_1p_{11}p_{10}$$

respectively. Then we gather them as follows:

$$00 + 010 \rightarrow 010 + 0010 + 0011, \quad 10 + 110 \rightarrow 110 + 1010 + 1011.$$

The following simple calculation checks the correctness of this construction:

$$\begin{aligned} K([00]) + K([010]) &= p_0(p_{00} + pp_{01}p_{10}) = p_0(1 - p_{11}) = p_0p_{10} \\ K([0010]) + K([0011]) + K([010]) &= p_1p_{10}^2(\bar{p}p_{00} + \bar{p}p_{01} + p) \\ &= p_1p_{10}^2 = p_0p_{01}p_{10} = (p_0p_{10})p_{01}. \end{aligned}$$

□

Theorem 7 *The process $\pi(p, v = 1, r = 1/2, \mathbb{Z}) \quad \forall v \in \mathbb{Z}_+^1, p, \varrho \in (0, 1)$ possesses a 1-parameter family of probabilistic invariant measures $\{\mu_{p, 1/2, 1, \mathbb{Z}, \varrho}\}_{\varrho}$ positive on each open set and supported by the set of configurations of density ϱ .*

Proof. Observe that the probability measure μ constructed in Lemma 4 due to the additional condition (4.2) can be parametrized by a single parameter $a := p_{01} \in (0, 1)$. From (4.2) we obtain

$$p_{10} = (1 - a)/(1 - pa). \quad (4.4)$$

In view of the equality between the density of particles ϱ and the stationary probability of ones p_1 under the Markov shift with the transition matrix P we have

$$\varrho = a/(a + p_{10}) = a(1 - pa)/(1 - pa^2).$$

Solving the last equality with respect to the parameter a , we get

$$a = \frac{1 - \sqrt{1 - 4p\varrho(1 - \varrho)}}{2p(1 - \varrho)}. \quad (4.5)$$

Therefore, for a given p the constructed family of measures $\{\mu_\varrho^{(p)}\}_\varrho$ is uniquely indexed by the density ϱ .

The property that the measures $\{\mu_\varrho^{(p)}\}_\varrho$ are massive follows immediately from the observation that the measure of an arbitrary finite cylinder is positive. \square

4.2 Existence of massive invariant measures in continuum

Existence of massive $\pi(0 < p < 1, v = 1, r = 1/2, \mathbb{Z})$ -invariant measures $\mu_\varrho^{(p)}$ for each ϱ is already proven in Section 4.1.

By Lemma 1(1) the measures $\mu_\varrho^{(p)}$ are $\pi(p, v = 1, r = 1/2, \mathbb{R})$ -invariant. For each $r' \geq 0$ applying the affine transformation (2.1), obtained in Lemma 1(3), to the measure $\mu_\varrho^{(p)}$ we are getting a new probability measure $\mu_{\varrho', r'}$ with $\varrho' := \varrho/(1 + 2(1/2 - r')\varrho)$, which is $\pi(p, v = 1, r', \mathbb{R})$ -invariant. Since ϱ takes all values from the interval $[0, 1]$ the new variable ϱ' takes all values from the interval $[0, 1/(2r')]$.

Making yet another spatial change of variables $z \rightarrow vz + w$ with parameters $v > 0$, $w \geq 0$ we are obtaining from the measure $\mu_{\varrho', r'}$ a two-parameter family of probability measures $\mu(p, \varrho'', r'', v, w)$ supported on configurations of balls of radius $r'' := vr'$ and having density $\varrho'' := \varrho'/v$. Applying again Lemma 1(4) we see that for each ϱ'', r', v, w the measure $\mu(p, \varrho'', r'', v, w)$ is $\pi(p, v, r'', \mathbb{R})$ -invariant.

On the other hand, these measures cannot be massive, since they are supported by very thin sets. To construct a massive (albeit non-ergodic) invariant measure from this family we consider a measure $\mu(p, \varrho'', r'', v)$ having marginals $\mu(p, \varrho'', r'', v, w)$ on sublattices $v\mathbb{Z} + w$ and uniformly distributed with respect to the parameter $w \in [0, v)$.

The construction of the massive invariant measure for $\pi(p, v, r, \mathbb{Z})$ with general $v \in \mathbb{Z}_+$ is exactly the same except that $w \in \{0, 1, \dots, v - 1\}$ and $r'' = 1/2$ or $r'' = 0$.

4.3 Stochastic stability

Despite very substantial differences in the constructions of nontrivial invariant measures in deterministic and random situations the structure of the invariant measure in both cases is described in terms of 2×2 Markov matrices $\{p_{ij}^{(p)}\}$. Therefore to prove the stochastic stability one only needs to check that

$$p_{ij}^{(p)} \xrightarrow{p \rightarrow 1} p_{ij}^{(1)}.$$

To this end one uses explicit formulas for the entries of the Markov matrices. Namely by (4.4)

$$p_{10}^{(p)} = (1 - a)/(1 - pa) \xrightarrow{p \rightarrow 1} 1,$$

which coincides with the corresponding entry in the deterministic case, while $p_{11}^{(p)} \xrightarrow{p \rightarrow 1} 0$. Passing to the limit as $p \rightarrow 1$ one gets the corresponding limit relations for two other

entries p_{00}, p_{01} , which define uniquely the Markov measure in the lattice deterministic setting with $v = 1$. Thus the nontrivial invariant measures for the process $\pi(p = 1, v = 1, r = 1/2, \mathbb{Z})$ are stochastically stable.

Recall now that the nontrivial invariant measures for the exclusion type processes in continuum $\pi(p, v, r, \mathbb{R})$ were constructed in Sections 3.2, 4.2 in three steps through changes of variables which do not depend on the choice of the variable p . Therefore this construction withstand the limit transition as $p \rightarrow 1$, which proves the stochastic stability for the general process $\pi(p = 1, v, r, \mathbb{R})$. Similarly one proves the stochastic stability for the lattice processes with long jumps $\pi(p = 1, v \in \mathbb{Z}_+, r \in \{0, 1/2\}, \mathbb{Z})$.

Let me note that the Markov measures are unique among stochastically stable translationally invariant measures, but there are non-translationally invariant ones. To demonstrate this consider a δ -measure μ supported by a single configuration $x := (\dots, 0, x_0 = 0, x_1 = 1, 1, \dots)$. Then this configuration is a fixed point for the process $\pi(p, v = 1, r = 1/2, \mathbb{Z})$ for each $0 < p \leq 1$ and hence the measure μ is stochastically stable.

5 Average velocities

Explicit expression for the invariant measure μ allows one to derive by Birkhoff's ergodic theorem the formula for the average velocity for μ -a.a. initial configurations. In what follows we are interested in a much more broad setting of all particle configurations having well defined densities. This extension can be justified as follows. Theorem 3 shows that for all particle configurations of a given density the average velocity is the same (or for all of these configurations the average velocity is not well defined). Therefore it is enough to calculate this statistics for a most suitable single initial configuration of given density. Choosing a configuration typical with respect to the invariant measure μ we achieve this goal. Since this construction does not depend on specific properties of the invariant measure we shall not repeat this argument in further calculations.

5.1 Lattice TASEP with $v = 1$

Let us use the constructed Markov measures to calculate the average velocities for the process $\pi(p, v, r = 1/2, \mathbb{Z})$.

Lemma 5 $V(\varrho) = (1 - \sqrt{1 - 4p\varrho(1 - \varrho)})/(2\varrho)$ for $\pi(p < 1, v, r = 1/2, \mathbb{Z})$.

Proof. For a given configuration of particles, a particle may move iff the next site of the lattice is not occupied, i.e. only in the situation 10. Therefore the average velocity is equal to the stationary probability of the jump to the right, which in turn is equal to pp_{10} . The representation of the Markov invariant measure obtained earlier immediately gives the formula for the average velocity

$$V(\varrho, p, 1, 1/2) = pp_{10} = p(1 - a)/(1 - pa) \in [0, p].$$

Substituting the value $a = a(\varrho)$ according to the formula (4.5) we get

$$V(\varrho, p, v = 1, r = 1/2) = \left[1 - \sqrt{1 - 4p\varrho(1 - \varrho)}\right] / (2\varrho). \quad (5.1)$$

□

5.2 Process in continuum

By means of results of Theorem 3 and Lemma 1 the relation (5.1) can be transferred to the processes of the 3d type $\pi(p, v, r, \mathbb{R})$ without changes. Note that the formula (5.1) was already known in physics publications for the lattice processes of type 1 with $v = 1$ (see [22]).

It is important to say that the naive transition from $v = 1$ to $v > 1$ directly in the class of lattice processes (using the invariance of sub-lattices with step multiple to v) is impossible (or rather so we can study only low-density $< 1/v$ configurations). Instead, we use the self-similarity of processes of type 3 acting in a continuous space.

Lemma 6 *Let the $\pi(p, v, r, \mathbb{R})$ type processes x^t, \acute{x}^t with parameters $r = \acute{r} = 0$, $v > 0$, $\acute{v} = uv$, $p = \acute{p} \in (0, 1)$ having initial configurations $\acute{x}^0 = ux^0$ with some $u > 0$ be statically coupled. Assume also that $\varrho(x)$ and $V(x)$ be well defined. Then*

$$\varrho(\acute{x}) = \varrho(x)/u, \quad \acute{V}(\acute{x}) = uV(x). \quad (5.2)$$

Proof. By a straightforward calculation. □

Applying these similarity transformations to the special case described in the relation (5.1) we obtain the general formula for the average velocity (1.3).

Corollary 7

$$\begin{aligned} V(\varrho, p, v > 0, r = 0) &= vV(v\varrho, p, v = 1, r = 1/2) \\ &= \frac{1 + v\varrho - \sqrt{(1 + v\varrho)^2 - 4pv\varrho}}{2\varrho} \\ &\xrightarrow{p \rightarrow 1} \min(v, 1/\varrho). \end{aligned}$$

Since the last term above coincides with the average velocity for the deterministic traffic map (see [7]), the limit transition in the last relation shows that the average velocity is stochastically stable.

Indeed, under a spatial change of variables $1 \rightarrow v$ and the transition from the configuration of balls of radius $r = 1/2$ to the configuration of point-particles (i.e. $\acute{r} = 0$) with the same sequence of gaps we get

$$\acute{\varrho} = \varrho/(1 - 2r\varrho) = \varrho/(1 - \varrho),$$

hence $\varrho = \acute{\varrho}/(1 + \acute{\varrho})$ and

$$\begin{aligned} V(\acute{\varrho}, p, v = 1, \acute{r} = 0) &= \frac{1 - \sqrt{1 - 4p\acute{\varrho}(1 - \acute{\varrho})}}{2\acute{\varrho}} \\ &= \frac{1 - \sqrt{1 - 4p\frac{\acute{\varrho}}{1+\acute{\varrho}}(1 - \frac{\acute{\varrho}}{1+\acute{\varrho}})}}{2\frac{\acute{\varrho}}{1+\acute{\varrho}}} \\ &= \frac{1 + \acute{\varrho} - \sqrt{(1 + \acute{\varrho})^2 - 4p\acute{\varrho}}}{2\acute{\varrho}}. \end{aligned}$$

Therefore by Lemma 6

$$\begin{aligned}
V(\varrho, p, v > 0, r = 0) &= vV(v\varrho, p, v = 1, r = 1/2) \\
&= \frac{1}{v} \times \frac{1 + v\varrho - \sqrt{(1 + v\varrho)^2 - 4pv\varrho}}{2v\varrho} \\
&= \frac{1 + v\varrho - \sqrt{(1 + v\varrho)^2 - 4pv\varrho}}{2\varrho} \xrightarrow{p \rightarrow 1} \min(v, 1/\varrho).
\end{aligned}$$

This finishes the proof of Theorem 2.

5.3 Lattice exclusion processes with long jumps

The results for $\pi(p, v, r, \mathbb{R})$ with arbitrary $v \in \mathbb{Z}_+$ are transferred directly by Lemma 1 back to the lattice cases. Indeed, Theorem 3 shows that it is enough to derive the formula for the average velocity for a specially chosen initial configuration of a given density. On the other hand, by Lemma 1(1 and 2) a realization of the processes $\pi(p, v, r = 1/2, \mathbb{Z})$ and $\pi(p, v, r = 0, \mathbb{Z})$ starting from certain configurations coincide with a realization of the $\pi(p, v, r, \mathbb{R})$ process statically coupled to the lattice process and starting from the same initial configuration.

It is worth noting that a naive application of the property described in Lemma 1(4) seems to extend the results about the process $\pi(p, v = 1, r = 1/2, \mathbb{Z})$ directly to $\pi(p, v > 1, r = 1/2, \mathbb{Z})$ restricting the latter process to invariant sub-lattices $v\mathbb{Z} + w$, $w = 0, 1, \dots, v-1$. A close look shows that this is indeed the case but only for configurations of low density $\varrho < 1/v$, since otherwise particles from the same configuration located at different sub-lattices will interact.

5.4 Heterogeneous particles (of different sizes)

Thinking about the processes under consideration as models of traffic flows it is reasonable to take into account that vehicles need not to be of the same size. From this point of view we consider an exclusion type process in continuum with particle configurations consisting of balls with varying sizes, i.e. the radius of the i -th ball is equal to $r_i \geq 0$. Our aim is to show that the dependence of the average velocity on density in this case can be easily obtained from the corresponding result for the case of balls of the same radius.

Let x be a bi-infinite admissible configuration of particles represented by balls of radiuses $r_i \geq 0$ with the average value

$$\bar{r} := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} r_i$$

and centered at points $x_i \in \mathbb{R}$. The notion of admissibility and the law of dynamics should be slightly rewritten (in comparison to the homogeneous case):

$$\begin{aligned}
x_i + r_i &\leq x_{i+1} - r_{i+1}, \\
x_i^{t+1} &= \begin{cases} \min\{x_i^t + v, x_{i+1}^t - r_i - r_{i+1}\} & \text{with probability } p \\ x_i^t & \text{with probability } 1 - p \end{cases} .
\end{aligned}$$

Theorem 8 *Let the process x^t defined above and the $\pi(p, v, \acute{r}, \mathbb{R})$ process \acute{x}^t be statically coupled and let $\acute{r} = \bar{r}$. Then the average velocities of these processes coincide.*

Proof. A simple generalization of the affine conjugation between configurations of different ball's sizes introduced in Lemma 1 allows one to make the bijection between the configurations x^t, \acute{x}^t . Indeed, consider an affine map defined by the relation

$$(\varphi(\acute{x}))_i := x_i - 2 \sum_{j=0}^i (r_j - \acute{r}).$$

Then

$$\begin{aligned} \frac{1}{n}(x_{n-1}^t - x_0^t) &= \frac{1}{n}((\varphi^{-1}(\acute{x}^t))_{n-1} - (\varphi(\acute{x}^t))_0) \\ &= \frac{1}{n}(\acute{x}_{n-1}^t - \acute{x}_0^t) + \frac{2}{n} \sum_{j=0}^{n-1} (r_j - \acute{r}) - \frac{2}{n}(r_0 - \acute{r}) \\ &\xrightarrow{n \rightarrow \infty} 1/\varrho(\acute{x}^t) + 2(\bar{r} - \acute{r}). \end{aligned}$$

Thus $\varrho(x^t) = \varrho(\acute{x}^t)$ if $\acute{r} = \bar{r}$.

Now we are ready to calculate the average velocity $V(x)$:

$$\begin{aligned} V(x, i, t) &= \frac{1}{t}(x_i^t - x_i^0) = \frac{1}{t}((\varphi^{-1}(\acute{x}^t))_i - (\varphi(\acute{x}^0))_i) \\ &= \frac{1}{t}(\acute{x}_i^t - \acute{x}_i^0) + \frac{2}{t} \sum_{j=0}^i (r_j - \acute{r}) - \frac{2}{t}(r_0 - \acute{r}) \\ &\xrightarrow{t \rightarrow \infty} V(\acute{x}) + 2(\bar{r} - \acute{r}), \end{aligned}$$

which proves our claim. \square

5.5 Heterogeneous space

So far we have considered only exclusion processes acting on homogeneous spaces. In [8] we introduced and studied a modification of the deterministic version of the exclusion process in continuum which takes into account the presence of static obstacles (traffic lights) for the motion of point particles (i.e. $r = 0$). Fix an arbitrary point-particle configuration $z = (z_j)_{j \in \mathbb{Z}} \in X(0, \mathbb{R})$ whose elements correspond to positions of obstacles. Then the formula (1.1) can be rewritten as follows:

$$x_i^{t+1} = \begin{cases} \min\{x_i^t + v, x_{i+1}^t, z_{j(x_i^t)}\} & \text{with probability } p \\ x_i^t & \text{with probability } 1 - p \end{cases}, \quad (5.3)$$

where $j(x_i^t) := \min\{k \in \mathbb{Z} : x_i^t \leq z_k\}$. Thus the ‘‘obstacles’’ suspend the movement of particles, taking into account the time necessary to overtake an obstacle.

For a given $v > 0$ and a configuration of obstacles z denote by \tilde{z} the *extended* configuration of obstacles obtained by inserting between each pair of entries z_i, z_{i+1} new $\lfloor (z_{i+1} - z_i)/v \rfloor$ ‘virtual’ obstacles at distances v between them starting from the point z_i . Here $\lfloor u \rfloor$ stands for the integer part of the number u .

Theorem 9 *For given $v > 0 < p < 1$ and any configurations $x, z \in X$ for which the densities $\varrho(x), \varrho(\tilde{z})$ are well defined*

$$V(x, z) = \frac{\varrho(x) + \varrho(\tilde{z}) - \sqrt{(\varrho(x) + \varrho(\tilde{z}))^2 - 4p\varrho(x)\varrho(\tilde{z})}}{2\varrho(x)\varrho(\tilde{z})} \xrightarrow{p \rightarrow 1} \min\{1/\varrho(\tilde{z}), 1/\varrho(x)\}. \quad (5.4)$$

Note that the average velocity in the above formula does not depend explicitly on the local velocity v , however the latter is included to the construction of the extended configuration \tilde{z} , in particular $\varrho(\tilde{z}) \geq 1/v$. It is interesting to note also that in [8] it was shown that under more general setting with the non-degenerate distribution of random iid local velocities v_i^t the average particle velocity may not exist.

The difficulty of the analysis here is that the inhomogeneity of the space in which the collective random walk takes place (the presence of obstacles), generally does not permit the existence of invariant measures.⁵ Therefore, the main step of the approach we used – the construction of a massive invariant measure is impossible. The complete proof of this result needs a modification of the dynamical coupling construction elaborated in [7, 8] and will be discussed elsewhere. Here we only give an idea of the proof.

Using the technique developed in [8] for the deterministic version of the problem it can be shown that the calculation of average velocities $V(x, z, v, p)$ can be reduced to the analysis of a Markov process of type 2, acting (in contrast to the already-studied setting) on *a inhomogeneous lattice* $\mathbf{R} := \tilde{z}$. The existence of the density of the configuration \tilde{z} allows one to transfer the results obtained for the conventional integer lattice \mathbb{Z} to the inhomogeneous case under consideration. This completes the construction.

⁵For the existence of invariant measures one needs at least the condition of stationarity for the configurations of obstacles z .

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